Calculus of Variations
Mastermath M1

Universiteit Utrecht
Spring 2020
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Course description

1) Prerequisites

Real Analysis, Measure Theory, Functional Analysis, in particular, knowledge of

- Banach and Hilbert spaces, convergence in these spaces
- Function spaces: bounded functions, continuous functions (with compact support), differentiable functions
- Measurable spaces and measures (Lebesgue, Borel)
- Measure spaces and measurable functions
- Lebesgue integral and L^p-spaces
- Convergence criteria: Vitali convergence theorem, monotone convergence theorem, dominated convergence theorem, Fatou's lemma
- Convex sets and functions, Jensen's inequality
- Notion of (relative) compactness
- Linear operators: definition and basic properties
- Dual spaces

The necessary background on these topics can be found in Chapters 2-5 in the book by Hans Wilhelm Alt, *Linear Functional Analysis: An Application-Oriented Introduction*, Springer.

Master-level courses on partial differential equations and functional analysis will be helpful, but are not mandatory.

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2) Aim of the course

The calculus of variations is an active area of research with important applications in science and technology, e.g. in physics, materials science or image processing. Moreover, variational methods play an important role in many other disciplines of mathematics such as the theory of differential equations, optimization, geometry, and probability theory.

The goal of this course is to give an introduction to different facets of this interesting field, which is concerned with the minimization (or maximization) of functionals.

By the end of the course, the student should be able to

- derive *variational models* to describe real world situations
- deduce *Euler-Lagrange equations* via the *first variation*
- exploit methods from the *theory of differential equations* to identify explicit representations or other properties (e.g. regularity) of solutions to variational problems
- apply the *direct method in the calculus of variations* to prove existence of minimizers

- deduce *lower semicontinuity of integral functionals* based on convexity properties of the integrands
- recall the notions of *generalized convexity* (poly-, quasi- and rank-one convexity) and their relations
- identify non-existence of solutions to variational problems and use *relaxation theory* to characterize behavior of almost minimizers
- apply *Young measure techniques* to identify oscillatory behavior of minimizing sequences
- characterize the asympototic behavior of parameter-dependent variational problems via Γ-*convergence*

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3) Rules about Homework/Exam

The course counts for 8 EC. The learning goals will be assessed as follows:

- hand-in assignments (15%)
- presentation of homework in class (5%)
- student project with poster session (10%)
- oral exam (70%)

The hand-in assignments, presentation of homework in class, and the student project with poster session still count as part of the grade after retake.

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4) Literature: The primary reference for this course is the book "Calculus of Variations" by F. Rindler, Universitext, Springer, 2018. For further reading, we recommend also

[1] H. Attouch, G. Buttazzo and G. Michaille, *Variational analysis in Sobolev and BV spaces - Applications to PDEs and optimization.* SIAM, Philadelphia, 2014.

[2] B. Dacorogna, *Direct methods in the calculus of variations.* Springer, New York, 2008.

[3] L.C. Evans, *Partial differential equations.* American Mathematical Society, Providence, RI, 1998.

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5) Lecturers:

- Dr. Carolin Kreisbeck (UU), email: c.kreisbeck@uu.nl (week 6-13)
- Dr. Oliver Tse (TU/e), email: o.t.c.tse@tue.nl (week 14-21)